Linear System Fundamentals

MEM 355 Performance Enhancement of Dynamical Systems

Harry G. Kwatny
Department of Mechanical Engineering & Mechanics
Drexel University

Contents

- System Representation
- Stability Concepts
- Similarity Transforms & Canonical Forms
- Pole Placement via State Feedback
- Routh-Hurwitz Stability Criterion

Drexel
UNIVERSITY

System Representation

State Space & Transfer Function Representations

A linear time invariant system (LTI) can be represented by a system of first order differential equations:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

State Space or time domain model

$$sX(s) - x_0 = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

where $x \in R^n$ is the state, $u \in R^m$ is the (control) input $y \in R^p$ is the output

Take the Laplace transform to obtain and solve for $Y(s)$ in terms of $U(s)$

$$X(s) = [sI - A]^{-1} x_0 + [sI - A]^{-1} BU(s)$$

Transfer Function or Frequency domain model

$$Y(s) = C[sI - A]^{-1} x_0 + G(s)U(s), \quad G(s) = C[sI - A]^{-1} B + D$$

Drexel UNIVERSITY

Characteristic Equation

Recall, the characteristic equation of A is

$$\phi(\lambda) = |\lambda I - A| = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

The roots of $\phi(\lambda) = 0$ are the eigenvalues of A

The transfer function can be written

$$G(s) = C[sI - A]^{-1}B + D = C\frac{\mathrm{Adj}(A)}{|sI - A|}B + D = \frac{C\,\mathrm{Adj}(A)B + D|sI - A|}{|sI - A|}$$

So we see that in the SISO case

$$G(s) = \frac{n(s)}{d(s)}, \quad n(s) = C\,\mathrm{Adj}(A)B + D|sI - A|, \quad d(s) = \phi(s)$$

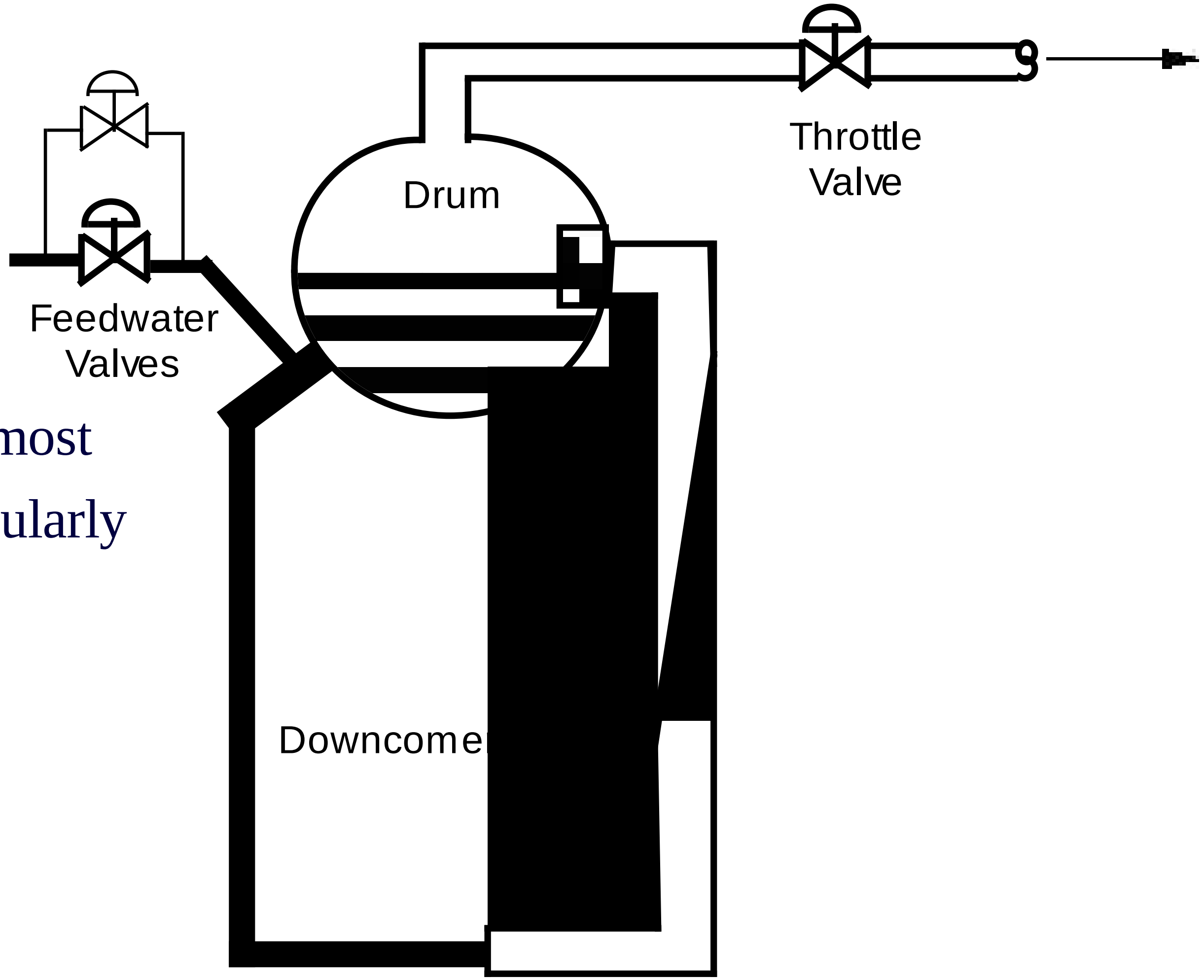
The poles of $G(s)$ are the eigenvalues of A!

Drexel UNIVERSITY

System Zeros

Example: Steam Power Plant Control

Inputs		Outputs	
heat flow	Q	drump pressure	P_d
water flow	ω_e	drum water level	ℓ
throttle valve	A_t	steam flow	ω_s



Traditional design used 3 SISO controllers. The most difficult issue was with drum level control, particularly at low load levels. Consider 3 options:

$$\omega_e \to \ell$$

$$\omega_e, A_t \to \ell, P_d$$

$$\omega_e, A_t, Q \to \ell, P_d, \omega_s$$

Power Plant, 2

N	number of riser sections
Ldo,L	downcomer length and riser section length (total riser length/N)
Ado,A	downcomer, riser cross section areas
wi	mass flow rate at ith node
Pi	pressure at ith node
Ti	temperature at ith node
si	aggregate entropy at ith node
vi	specific volume at ith node
wr,wdc,ws	mass flow rates, riser, downcomer and turbine, respectively
vdf,vdg	drum specific volume, liquid and gas, respectively
Pd	drum pressure
Td	drum temperature
V	total drum volume
Vw	volume of water in drum
xd	net drum quality, xd=Vw/V
ws0	throttle flow at rated conditions
Pd0	drum pressure at rated conditions
At	normalized throttle valve position, at rated conditions At=1

Power Plant, 3

$$u_1 = q,\ u_2 = \omega_e,\ u_3 = A_t$$

$$\frac{d\omega_{av}}{dt} = f_1(\omega_{av},s_1,s_2,s_3,P_{av},P_d)$$

$$\frac{ds_1}{dt} = f_2(\omega_{av},s_1,P_{av})+g_{21}(P_{av},s_1)u_1+g_{22}(\omega_{av},P_d)u_2$$

$$\frac{ds_2}{dt} = f_3(\omega_{av},s_1,s_2,P_{av})+g_{31}(P_{av},s_2)u_1$$

$$\frac{ds_3}{dt} = f_4(\omega_{av},s_2,s_3,P_{av})+g_{41}(P_{av},s_3)u_1$$

$$\frac{ds_4}{dt} = f_5(\omega_{av},s_3,s_4,P_{av})+g_{51}(P_{av},s_4)u_1$$

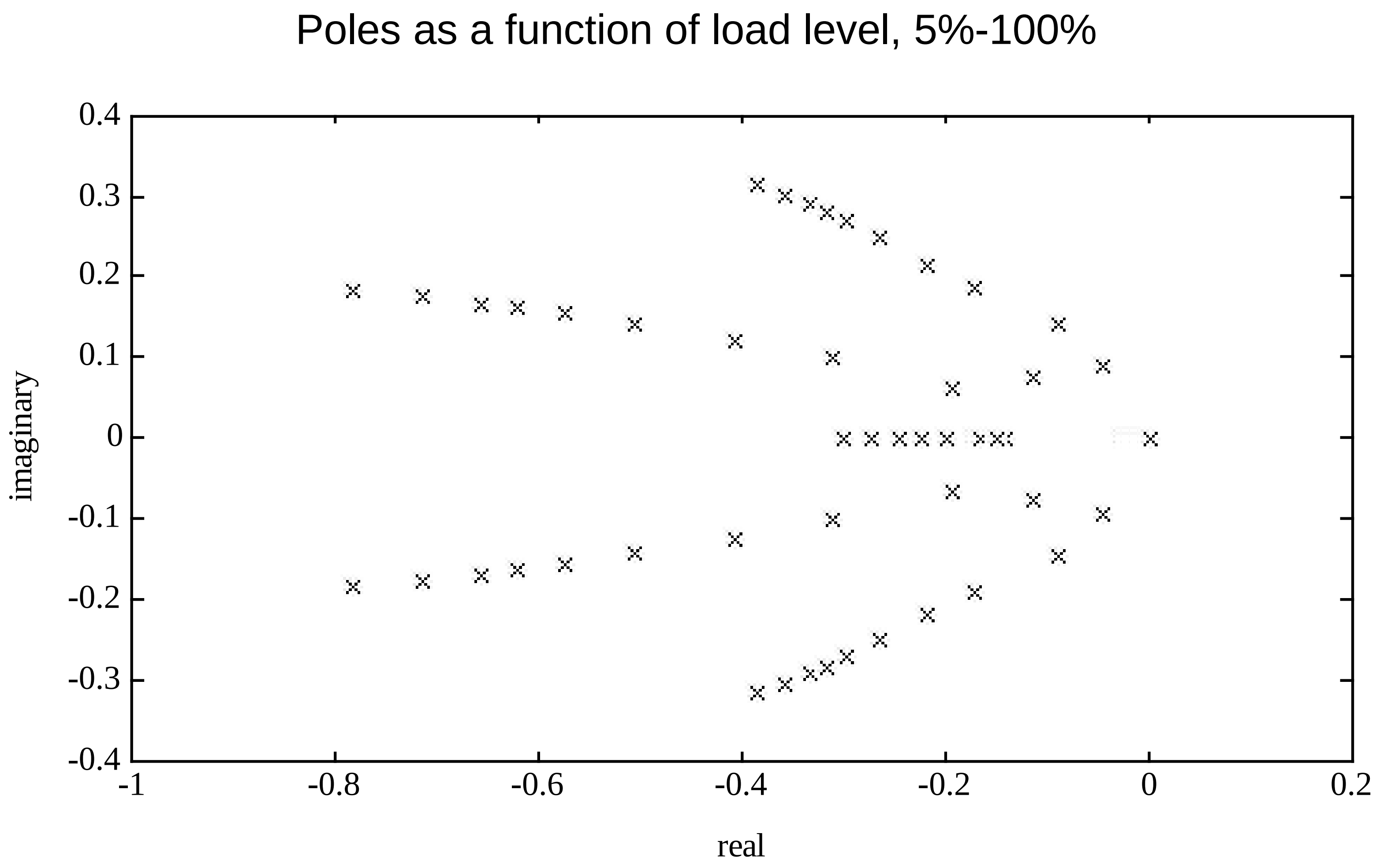
$$\frac{dP_{av}}{dt} = f_6(\omega_{av},s_1,s_2,s_3,s_4,P_{av},P_d)+g_{61}(\omega_{av},s_1,s_2,s_3,s_4,P_{av})u_1$$

$$\frac{dP_d}{dt} = f_7(\omega_{av},s_1,s_2,s_3,s_4,P_{av},P_d,V_w)+g_{71}(\omega_{av},s_1,s_2,s_3,s_4,P_{av},P_d,V_w)u_1+g_{72}(P_d,V_w)u_2-g_{73}(P_d,V_w)u_3$$

$$\frac{dV_w}{dt} = f_8(\omega_{av},s_1,s_2,s_3,s_4,P_{av},P_d,V_w)+g_{81}(\omega_{av},s_1,s_2,s_3,s_4,P_{av},P_d,V_w)u_1+g_{82}(P_d,V_w)u_2-g_{83}(P_d,V_w)u_3$$

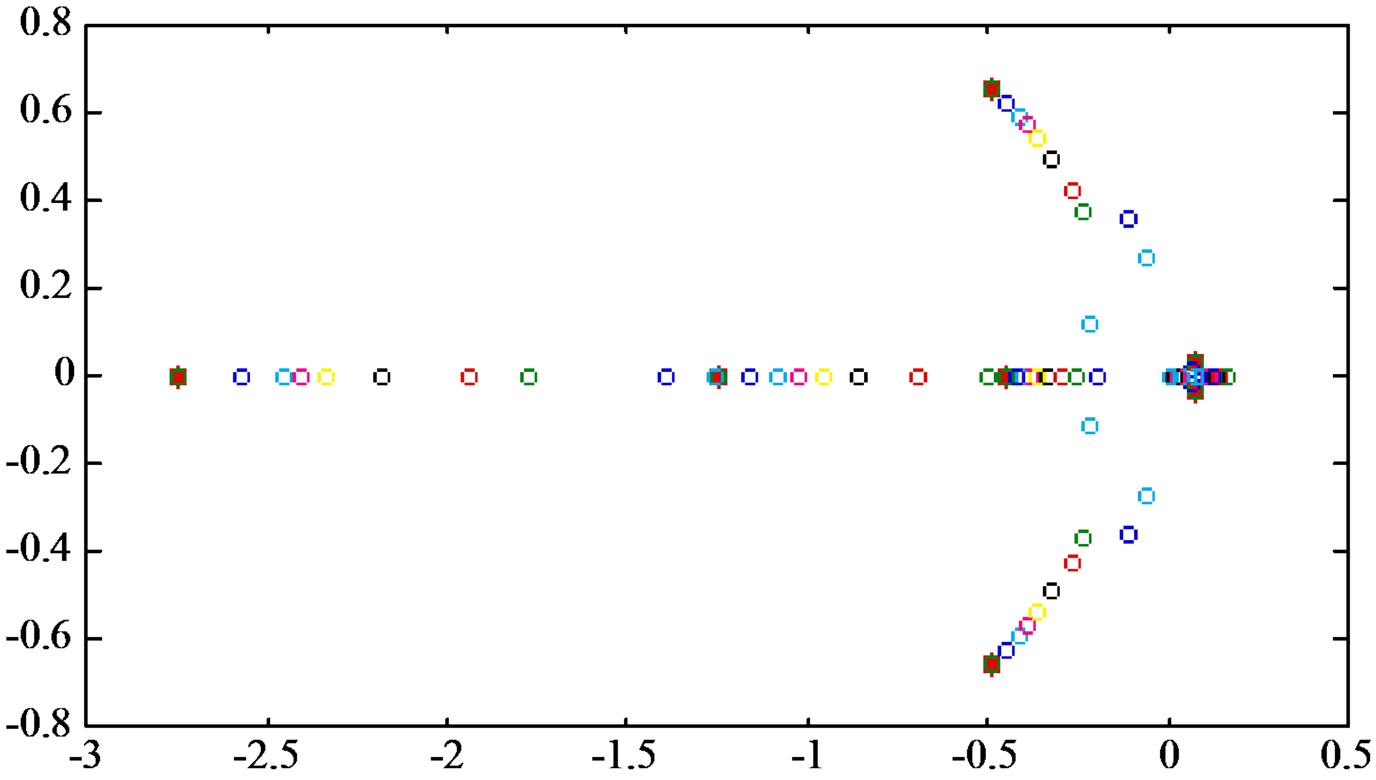
$$y_1 = P_d,\quad y_2 = ,\ = h_2(V_w),\quad y_3 = \omega_s = h_3(P_d)+d_3(P_d)u_3$$

Linearized Dynamics, Poles



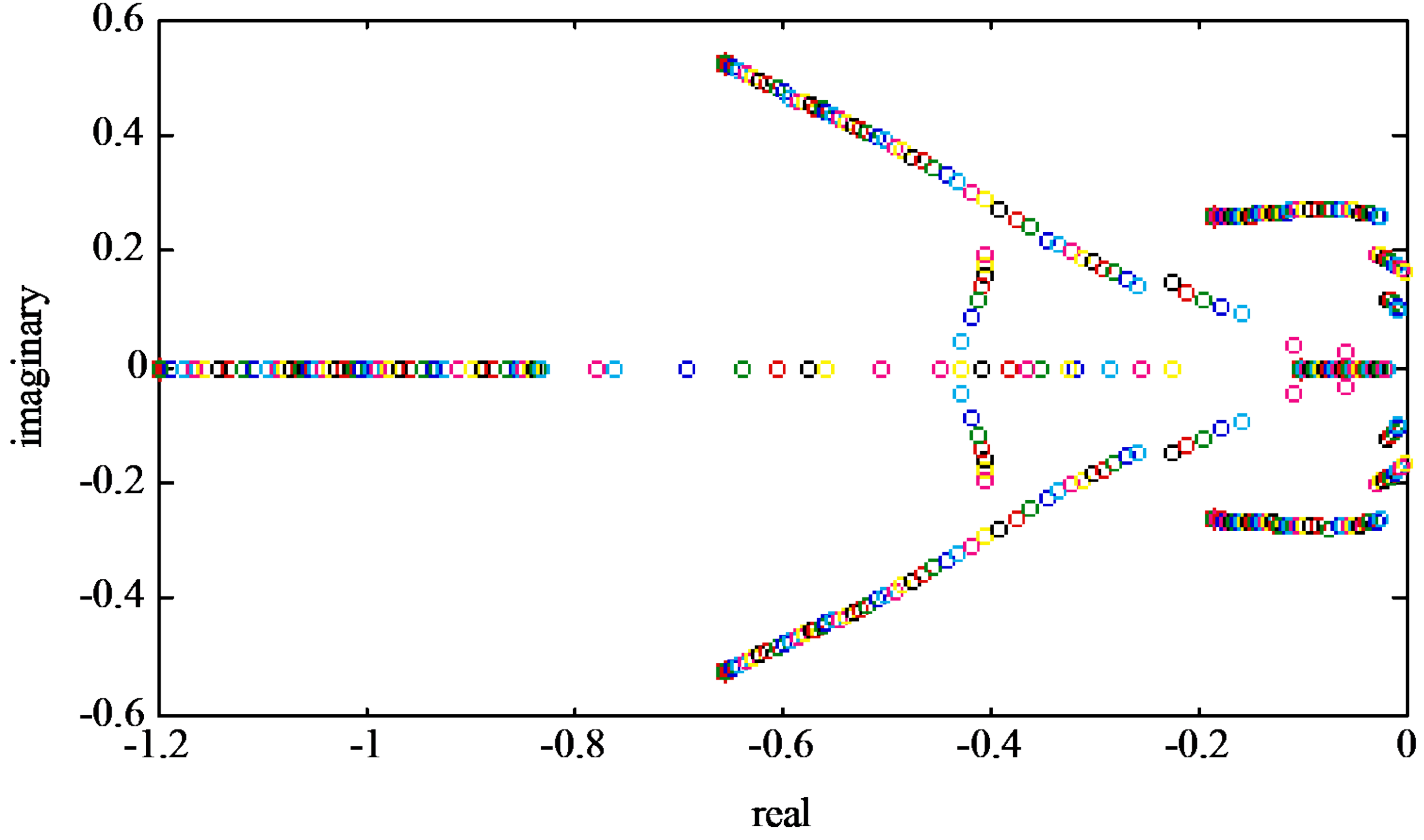
Linearized Dynamics, Zeros

$$\omega_e \rightarrow \ell$$



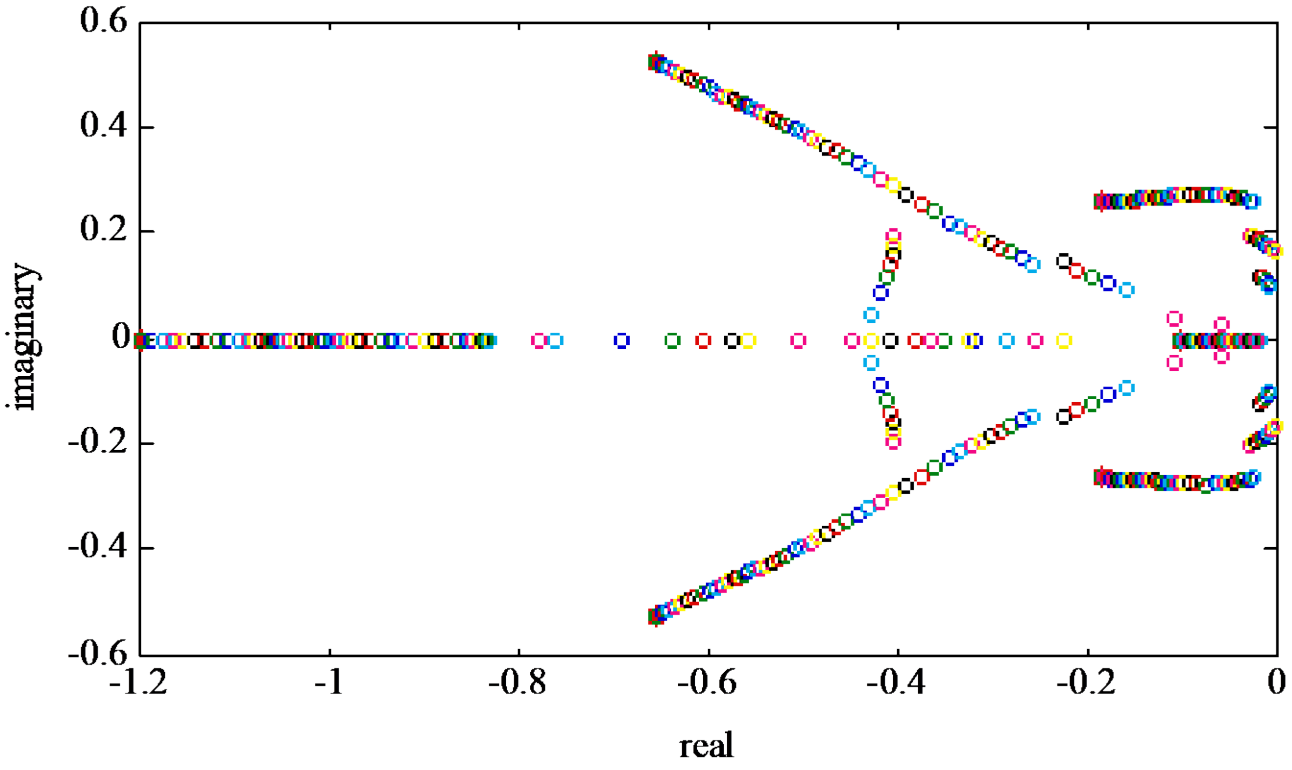
Transmission Zeros

$$\omega_e, A_t \rightarrow P_d, \ell$$



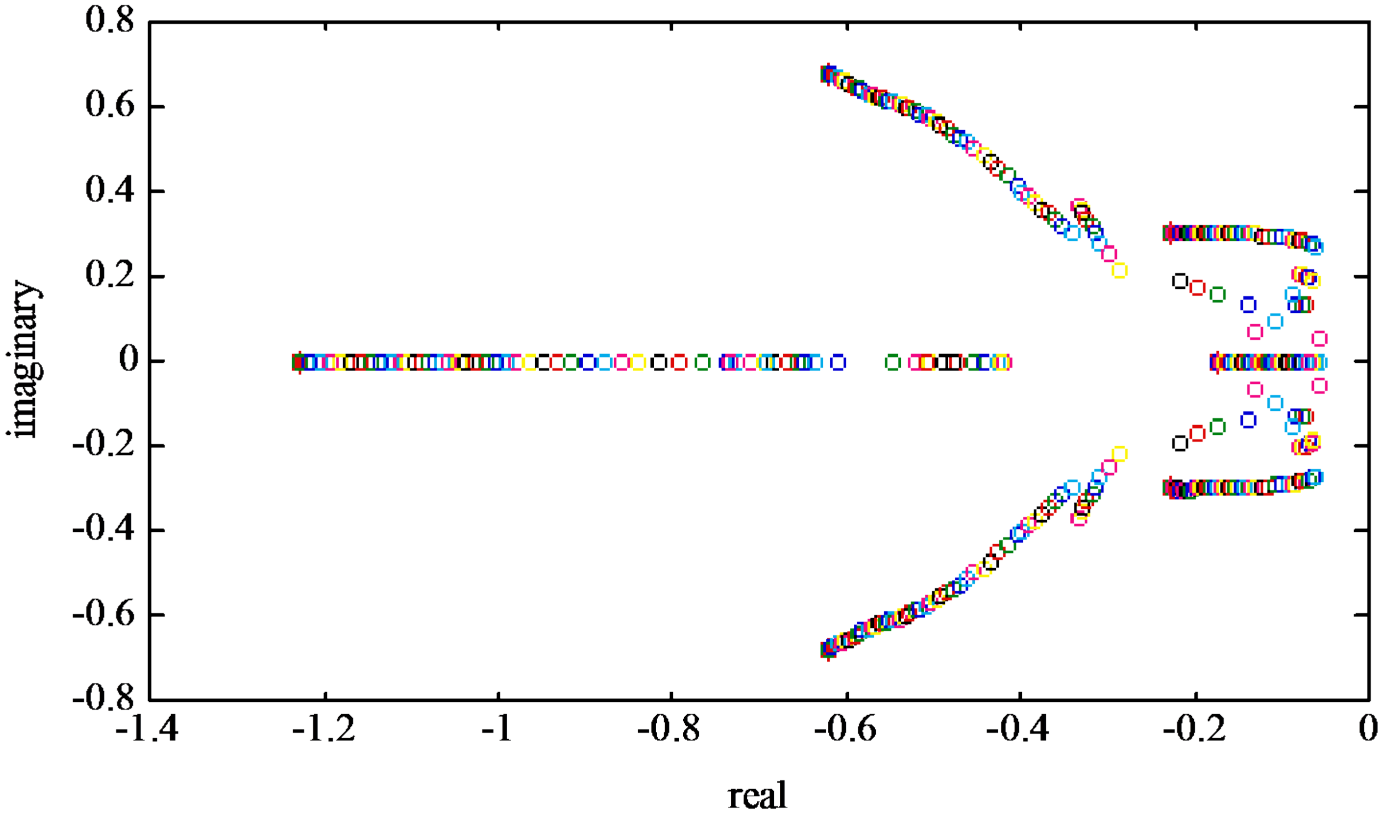
Transmission Zeros

$$\omega_e, A_t \rightarrow P_d, \ell$$



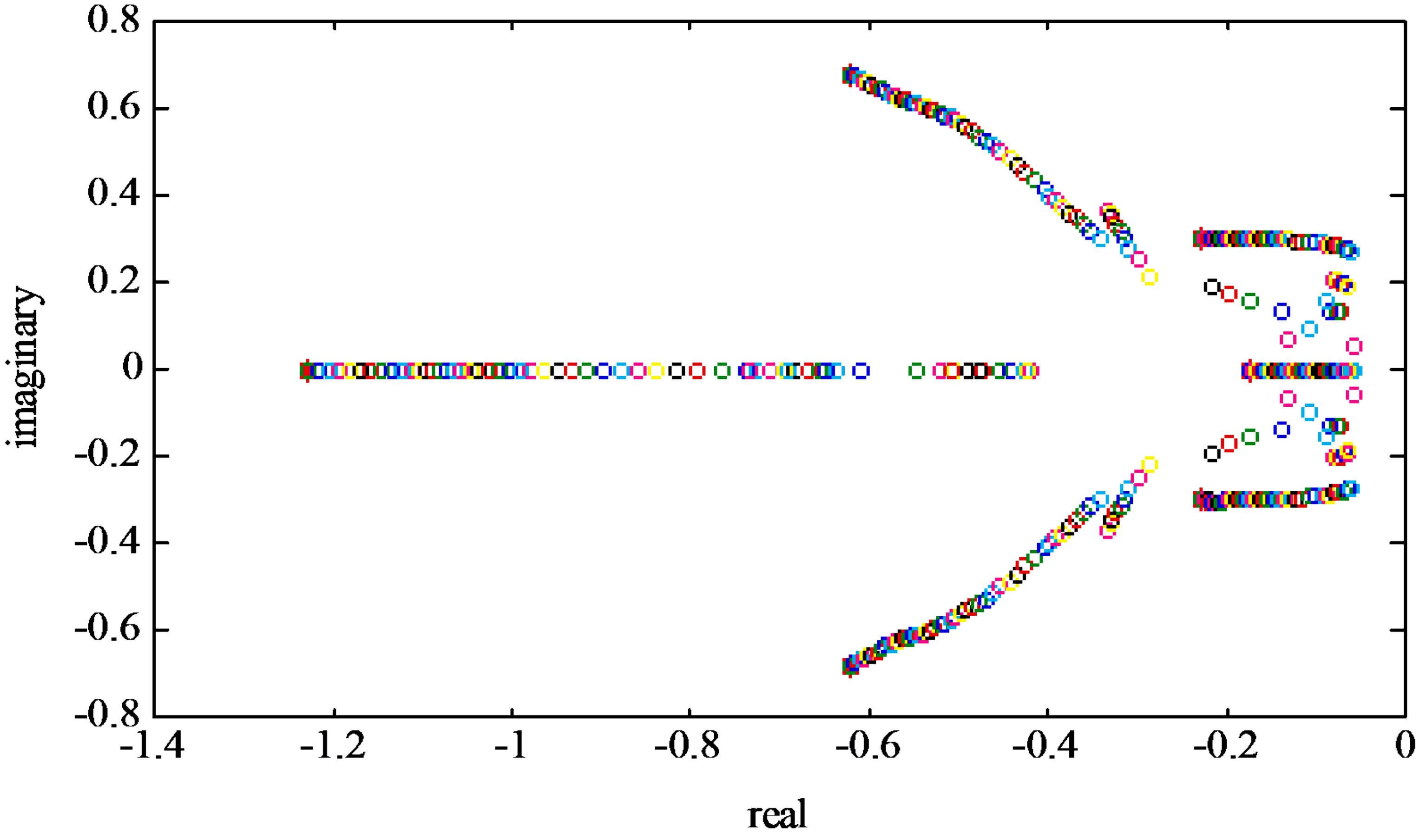
Transmission zeros

$$Q, \omega_e, A_t \rightarrow P_d, \ell, \omega_s$$



Transmission zeros

$$Q, \omega_e, A_t \rightarrow P_d, \ell, \omega_s$$



Resolvant Matrix and State Transition Matrix

The matrix $[sI - A]^{-1}$ is called the "resolvent" matrix.

It's inverse Laplace Transform is the "state transition matrix":

$$\Phi(t) = \mathcal{L}^{-1}\left([sI - A]^{-1}\right) = e^{At}$$

More about this later

In terms of Φ we can write

$$\mathrm{x}(t) = \Phi(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau$$

State response

$$y(t) = C\Phi(t)x_0 + C\int_0^t \Phi(t-\tau)Bu(\tau)d\tau + Du(t)$$

Output response

Drexel UNIVERSITY

A Stability Lemma

Lemma: Assume the system poles, i.e., the eigenvalues of the *A* matrix are all in the strict left complex plane, then we have the following:

(1) The response due to the initial state is,

$$\lim_{t\to\infty} \Phi(t)x_0 = 0$$

and

(2) The response to the input, $u(t)$ ($x_0 = 0$)

$$y(t) = C\int_0^t \Phi(t-\tau)Bu(\tau)d\tau + Du(t)$$

is bounded for every bounded input $u(t)$.

Drexel UNIVERSITY

Stability Definitions

Definition: A linear time-invariant system is <u>**BIBO**</u> (Bounded-Input Bounded-Output) stable if and only if every bounded input results in a bounded output.

Definition: A linear time-invariant system is <u>**internally**</u> stable if the solution $x(t)$ of

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0$$

- tends toward zero as $t \to \infty$ for arbitrary x_0 .

Aren't these the same? -------- No!

Drexel UNIVERSITY

Stability Theorems

Theorem: An LTI system with transfer function $G(s)$ is BIBO stable if and only if the poles of $G(s)$ are strictly in the left half plane.

Theorem: An LTI system with state space parameters A, B, C, D is internally stable if and only if the all of the eigenvalues of A are strictly in the left half plane.

Note: Internal stability is a stronger condition than BIBO stability. BIBO stability only reflects the attributes of the system that are **observable** from the output and **controllable** from the input. There may be hidden modes that are unstable.

Much more about this later

Similarity Transformations

$$\dot{x} = Ax + Bu \qquad \dot{x} = Ax + bu$$
$$y = Cx + Du \qquad x \in R^n, u \in R^m, y \in R^p \qquad y = cx + du$$

Now consider the transformation to new states z, defined by

$$x = Tz \Leftrightarrow z = T^{-1}x$$

$$\begin{aligned} T\dot{z} &= ATz + Bu \\ y &= CTz + Du \end{aligned} \Rightarrow \begin{aligned} \dot{z} &= T^{-1}ATz + T^{-1}Bu \\ y &= CTz + Du \end{aligned}$$

so that,

$$\begin{aligned} \dot{z} &= A^*z + B^*u \\ y &= C^*z + D^*u \end{aligned} \qquad A^* = T^{-1}AT,\ B^* = T^{-1}B,\ C^* = CT,\ D^* = D$$

Diagonal Form

eigen−system of A:

$$\begin{matrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n & \leftarrow \text{eigenvalues} \\ h_1 & h_2 & \cdots & h_n & \leftarrow \text{independent eigenvectors} \end{matrix}$$

$$T \triangleq [h_1 \quad h_2 \quad \cdots \quad h_n]$$

$$\Rightarrow A^* = [h_1 \quad h_2 \quad \cdots \quad h_n]^{-1} A [h_1 \quad h_2 \quad \cdots \quad h_n]$$

$$= \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

$$\boxed{\dot{z}_i = \lambda_i z_i + b_i^* u, \;\; i = 1, \dots, n}$$

A decoupled system of n 1st order ode's

Companion Form

Consider the single-input system:

$$\dot{x} = Ax + bu$$

and the transformation

$$\begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix}^{-1} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} q_n \\ q_n A \\ \vdots \\ q_n A^{n-1} \end{bmatrix}$$

Apply the similarity transform to obtain the system:

$$\dot{z} = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} z + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

Drexel UNIVERSITY

Proof (1)

We proceed in two steps: First establish b^*, then A^*

$$\mathcal{C}^{-1}\mathcal{C} = I \Rightarrow \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix} = I$$

$$\Rightarrow q_n \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$\Rightarrow b^* = T^{-1}b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Drexel UNIVERSITY

Proof (2)

$$T^{-1}T = I \Rightarrow \begin{bmatrix} q_n \\ q_n A \\ \vdots \\ q_n A^{n-1} \end{bmatrix} T = I \Rightarrow \begin{bmatrix} q_n T \\ q_n A T \\ \vdots \\ q_n A^{n-1} T \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$T^{-1}AT = \begin{bmatrix} q_n \\ q_n A \\ \vdots \\ q_n A^{n-1} \end{bmatrix} AT = \begin{bmatrix} q_n A T \\ q_n A^2 T \\ \vdots \\ q_n A^n T \end{bmatrix} = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \\ Y_1 & Y_2 & \cdots & Y_n \end{bmatrix}, Y = q_n A^n T$$

To compute Y, suppose $\det(\lambda I - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$.

note C-H Theorem $\Rightarrow A^n + a_{n-1}A^{n-1} + \cdots + a_0 I = 0$

$$Y = q_n A^n T = q_n\left(-a_{n-1}A^{n-1} - \cdots - a_0 I\right)T = -a_{n-1}q_n A^{n-1}T - \cdots - a_0 q_n T = \begin{bmatrix} -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix}$$

SISO Companion Forms

Controllability

$$\dot{z} = \begin{bmatrix} 0 & & 0 & -a_0 \\ 1 & \ddots & & -a_1 \\ & \ddots & 0 & \vdots \\ 0 & & 1 & -a_{n-1} \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$y = \bar{c} z$$

Observability

$$\dot{z} = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -a_{n-1} & \cdots & -a_1 & -a_0 \end{bmatrix} z + \bar{b} u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} z$$

Controller/phase variable

$$\dot{z} = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} z + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \bar{c} z$$

Observer

$$\dot{z} = \begin{bmatrix} -a_{n-1} & 1 & & 0 \\ \vdots & 0 & \ddots & \\ -a_1 & & \ddots & 1 \\ -a_0 & 0 & & 0 \end{bmatrix} z + \bar{b} u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} z$$

Drexel UNIVERSITY

State Feedback Pole Placement

Given a linear system:

$$\dot{x} = Ax + Bu$$

find a state feedback control:

$$u = Kx$$

such that the closed loop system:

$$\dot{x} = Ax + BKx = \left(A + BK\right)x$$

has a specified (self-conjugate) set of poles $\left\{p_1, p_2, \ldots, p_n\right\}$.

Drexel UNIVERSITY

Pole Placement Sol'n: SISO Case

- Convert $\dot{x} = Ax + bu$ to controller form (phase variable form) using $x = Tz$:

$$\dot{z} = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} z + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

← controller form

- Set $u = \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix} z$ and obtain closed loop: $\dot{z} = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & 0 & 1 \\ k_1 - a_0 & k_2 - a_1 & \cdots & k_n - a_{n-1} \end{bmatrix} z$

- Expand desired closed loop characteristic polynomial and compare coefficients, and solve for $k_1, \ldots, k_n$:

$$\phi_{cl}(\lambda) = (\lambda - p_1)(\lambda - p_2)\cdots(\lambda - p_n) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_0 \Rightarrow \alpha_0 = a_0 - k_1, \alpha_1 = a_1 - k_2, \ldots, \alpha_{n-1} = a_{n-1} - k_n$$

- Convert back to x-coordinates: $Kz = KT^{-1}x \Rightarrow u = \left(KT^{-1}\right)x$

Pole Place Design: The Easy Way

PLACE Pole placement technique

K = PLACE(A,B,P) computes a state-feedback matrix K such that
the eigenvalues of A-B*K are those specified in vector P.
No eigenvalue should have a multiplicity greater than the number of inputs.

Warning!! Notice the sign difference.

Example: XV-15 Hover Dynamics

$$\frac{d}{dt}\begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} = \begin{bmatrix} -0.0124 & -0.0025 & 1.3086 & -32.162 \\ 0 & -0.1901 & 0.1000 & 0.4963 \\ 0.0015 & 0.0003 & -0.15534 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 0.6264 & -0.0711 \\ 0 & -5.3507 \\ -0.0734 & 0.0075 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \delta_{loc} \\ \delta_{co} \end{bmatrix}$$

u, body x – velocity

w, body z – velocity

q, pitch rate

θ pitch angle

δ_{loc} rotor longitudinal cyclic pitch

δ_{co} collective pitch

Drexel UNIVERSITY

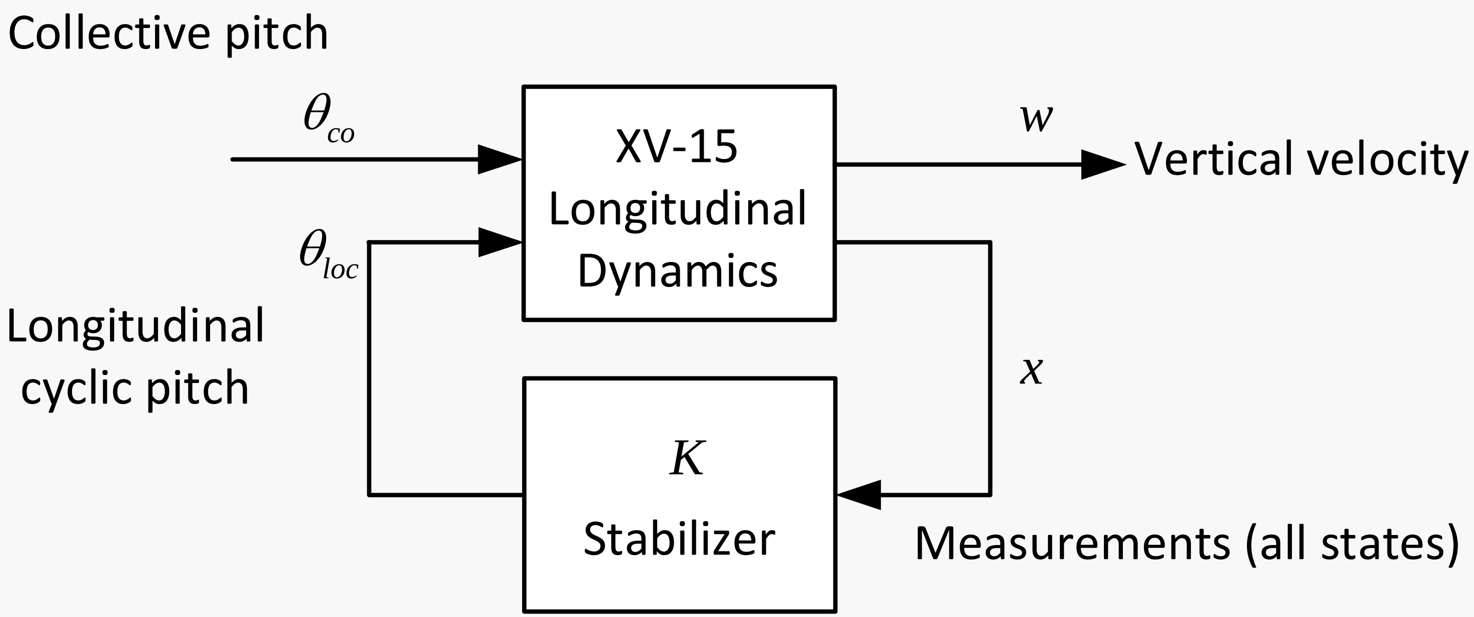
Example, XV15 Longitudinal Modes

unstable

λ	-0.4280	-0.1915	$0.1314 \pm j0.3084$
u	0.9996	0.2017	$0.9999 \pm j0.0000$
w	-0.0243	-0.9795	$-0.0110 \pm j0.0052$
q	-0.0054	-0.0002	$0.0024 \pm j0.0026$
θ	0.0127	0.0012	$-0.0044 \pm j0.0097$

Drexel UNIVERSITY

Example, XV-15 Longitudinal Stabilizer



Old eigenvalues:

$$p_{old} = \{-0.4280, -0.1915, -0.1314 \pm j0.3084\}$$

Choose new eigenvalues:

$$p = \{-0.5, -0.2, -0.25 \pm j0.25\}$$

XV-15 Pole Placement

PLACE Pole placement technique

K = PLACE(A,B,P)

$$A=\begin{bmatrix}-0.0124 & -0.0025 & 1.3086 & -32.162\\ 0 & -0.1901 & 0.1000 & 0.4963\\ 0.0015 & 0.0003 & -0.15534 & 0\\ 0 & 0 & 1 & 0\end{bmatrix},\quad b=\begin{bmatrix}0.6264\\ 0\\ -0.0734\\ 0\end{bmatrix},$$

$$p=\{-0.5,-0.2,-0.25+j0.25,-0.25-j0.25\}$$

$$K=\begin{bmatrix}0.0074 & 0.0236 & -11.4263 & -5.2175\end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix}u\\ w\\ q\\ \theta\end{bmatrix}=(A-bK)\begin{bmatrix}u\\ w\\ q\\ \theta\end{bmatrix}+\begin{bmatrix}-0.0711\\ -5.3507\\ 0.0075\\ 0\end{bmatrix}\theta_{co}$$

$$=\begin{bmatrix}-0.0170 & -0.0173 & 8.4660 & -28.8980\\ 0 & -0.1909 & 0.1000 & 0.4963\\ 0.0020 & 0.0020 & -0.9921 & -0.3830\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}u\\ w\\ q\\ \theta\end{bmatrix}+\begin{bmatrix}-0.0711\\ -5.3507\\ 0.0075\\ 0\end{bmatrix}\theta_{co}$$

$$\theta_{oc}\rightarrow w$$

$$G=-5.3507\frac{(s+0.5137)(s^2+0.4952s+0.1275)}{(s+0.5)(s+0.2)(s^2+0.5s+0.125)}$$

Drexel UNIVERSITY

Routh-Hurwitz Stability Criterian

- Given a polynomial that represents the pole polynomial of a transfer function or the characteristic polynomial of square matrix it is very easy to determine the roots (and therefore assess stability) using numerical computations – providing the coefficients are all specified!
- But suppose one or more of the coefficients are not specified. Rather, we want to determine an admissible range of values such that the system is stable. That is the control designer's problem and where the Routh-Hurwitz criterion is useful.

Routh-Hurwitz –2

Theorem: Consider the polynomial

$$p(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

A necessary condition that all roots are strictly in the left half plane is that all coefficients are strictly positive.

Note that this provides only a necessary condition. To determine a necessary and sufficient condition we assemble the Routh array.

Routh-Hurwitz–3

Theorem: Consider the polynomial

$$p(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

The associated system is (BIBO or internally) stable if and only if all elements of the first column of the Routh array are strictly positive.

If the polynomial is the 'minimal' transfer function denominator

If the polynomial is the Characteristic polynomial of the A matrix

Routh Array

$$p(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

$$\#\,\text{columns} = \begin{cases} \frac{n}{2} & n \text{ even} \\ \frac{n+1}{2} & n \text{ odd} \end{cases}$$

Construct a matrix as shown by with the first two rows using the using the above polynomial coefficients.

Then construct the remaining rows

$$b_1 = \frac{-\begin{vmatrix} 1 & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}}{a_{n-1}}, b_2 = \frac{-\begin{vmatrix} 1 & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}}{a_{n-1}}, \ldots$$

$$c_1 = \frac{-\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix}}{b_1}, c_2 = \frac{-\begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_2 \end{vmatrix}}{b_1}, \ldots$$

$$\longleftarrow - - - - - - - - \longrightarrow$$

$$\begin{array}{c|cccc} s^n & 1 & a_{n-2} & \cdots \\ s^{n-1} & a_{n-1} & a_{n-3} & \cdots \\ s^{n-2} & b_1 & b_2 & \cdots \\ s^{n-3} & c_1 & c_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ s^0 & \vdots & & \end{array}$$

Routh-Hurwitz – Example 1

$$p(s) = s^5 + 15s^4 + 74.25s^3 + 121s^2 + 20Ks + 2K$$

$$\begin{array}{c|lll}
s^5 & 1 & 74.25 & 20K \\
s^4 & 15 & 121 & 2K \\
s^3 & 65.9 & 19.86K & \\
s^2 & 121 - 4.52K & 2K & \\
s^1 & \dfrac{2271K - 89.76K^2}{121 - 4.52K} & & \\
s^0 & 2K & &
\end{array}$$

$$121 - 4.52K > 0,\ 2271K - 89.76K^2 > 0,\ K > 0$$

$$K < \frac{121}{4.52} = 26.769 \qquad \boxed{K < \frac{2271}{89.76} = 25.308}$$

Routh-Hurwitz – Example 2

$$p(s) = s^4 + (1+K)s^3 + (1+6K)s^2 + 10Ks + 8K$$

s^4	1	$1+6K$	$8K$
s^3	$1+K$	$10K$	0
s^2	$\dfrac{1-3K+6K^2}{1+K}$	$8K$	0
s^1	$\dfrac{2K(1-23K+26K^2)}{1-3K+6K^2}$	0	
s^0	$8K$		

$$1+K>0,\quad 1-3K+6K^2>0,\quad 1-23K+26K^2>0,\quad K>0$$